Mid Term 1

1\. (c) $\mathbb{R}^{(0,1)} = \{ f : (0,1) \to \mathbb{R} \}$ does have a basis

Reason: See Lecture Notes: Week 3. on Page 13

"Every Vector space has a basis."

This property needs advanced tools to prove. #

1\. (h). For $\mathbb{R}^\infty = \{ (x_1, x_2, \cdots) : x_i \in \mathbb{R} \text{ for all } i \in \mathbb{N} \}$,

$\{e_1, e_2, \cdots\}$ is NOT a basis,

where $e_k = (0, \cdots, 0, \underset{\substack{\uparrow \\ k\text{-th}}}{1}, 0, \cdots)$.

Reason: $\operatorname{span}(\{e_1, e_2, \cdots\})$

$$= \{ \underbrace{(x_1, \cdots, x_N, 0, \cdots)}_{\text{at most finite numbers are non-zero}} : x_i \in \mathbb{R},\ 1 \le i \le N,\ \forall N \ge 1 \}$$

but $(1, 1, \cdots, 1, \cdots) \in \mathbb{R}^\infty \setminus \operatorname{span}\{e_1, e_2, \cdots\}$. #

4\. V - vector space, $T : V \to V$ linear map.

(a). Show that $W = \{ v \in V : Tv = v \}$ is a subspace of V.

(b). Suppose $T^2 = T$. Show that $V = \operatorname{null} T \oplus W$.

Solution: (a) is easy to prove.

(b). $T^2 = T \Rightarrow \forall v \in V,\ T^2 v = Tv \Rightarrow T(v - Tv) = 0$

$\Rightarrow v - Tv \in \operatorname{Null} T$.

So $v = v - Tv + Tv \in \operatorname{Null} T + W$

If some $v \in \operatorname{Null} T \cap W \Rightarrow Tv = 0$ and $v = Tv$

$\Rightarrow v = 0$

So $V = \operatorname{Null} T \oplus W$. #

5. V - a real vector space, W_i - a subspace of V with $\dim W_i = 2$, $i=1,2$.

Suppose $W_1 \cap W_2 = \text{span}\{v_0\}$,

$W_1 = \text{span}\{v_0, v_1\}$, $W_2 = \text{span}\{v_0, v_2\}$

prove $\{v_0, v_1, v_2\}$ is linearly indept.

<u>Solution:</u> Assume $a_0 v_0 + a_1 v_1 + a_2 v_2 = 0$ for some $a_i \in \mathbb{R}$, $i=1,2,3$.

①. If $a_0 = a_1 = a_2 = 0$, then done.

②. If NOT, then $a_2 \neq 0$. Otherwise, $a_0 v_0 + a_1 v_1 = 0 \Rightarrow a_0 = a_1 = 0$.

So $v_2 = a_2^{-1} a_1 v_1 + a_2^{-1} a_0 v_0 \in \text{span}\{v_0, v_1\} = W_1$

$\Rightarrow v_2 \in W_1 \cap W_2 = \text{span}\{v_0\}$

$\Rightarrow \exists\ b \in \mathbb{R}$, s.t. $v_2 = b v_0 \Rightarrow$ Contradiction with v_0, v_2 linearly indept.

#

Week 5

Q1: (1). Suppose $\dim V = \dim W$ is finite, $T \in \mathcal{L}(V, W)$

Then T is injective $\Leftrightarrow$ T is surjective ($\Leftrightarrow$ T is bijective)

(2). If $\dim V = +\infty$, $T \in \mathcal{L}(V, V)$,

then T is injective (or surjective resp.) cannot imply that T is surjective (or injective resp.)

Proof: (1). By Thm 3.22, $\dim(\text{null } T) + \dim(\text{range } T) = \dim V$

"$\Rightarrow$": $\text{null } T = \{0\} \Rightarrow \dim(\text{null } T) = 0$

$\Rightarrow \dim(\text{range } T) = \dim V = \dim W$

while $\text{range } T$ is a subspace of W, then $\text{range } T = W$

$\Rightarrow T$ is surjective.

"$\Leftarrow$": similar proof as above.

(2). Take a basis of V: $\{e_n\}_{n=1}^{\infty}$

(a). Define $T \in \mathcal{L}(V, V)$

via: $T\left(\sum_{n=1}^{N} c_n e_n\right) = \sum_{n=1}^{N} c_n e_{2n}$

(i.e. Let $T(e_n) = e_{2n}$, $\forall n \geq 1$, and extend this mapping linearly to the whole space V).

Then T is injective.

(In fact, if $v = \sum_{n=1}^{N} c_n e_n \in \text{null } V$

then $Tv = \sum_n c_n e_{2n} = 0 \Rightarrow c_n = 0$, $\forall 1 \leq n \leq N$

$\Rightarrow v = 0$)

and $\text{range } T = \text{span}\{e_2, e_4, \cdots, e_{2n}, \cdots\} \neq V$

(b). Define $T \in \mathcal{L}(V, V)$ satisfying

$T(e_1) = 0$, $T(e_n) = e_{n-1}$, $\forall n \geq 2$.

$\left(T\left(\sum_{n=1}^{N} c_n e_n\right) \overset{\text{def}}{=\!=} \sum_{n=2}^{N} c_n e_{n-1}\right)$

Then $\text{range } T = V$

but $\text{null } T = \text{span}(\{e_1\}) \neq \{0\}$.

#

Q2:

13 Suppose T is a linear map from $\mathbf{F}^4$ to $\mathbf{F}^2$ such that

$$\operatorname{null} T = \{(x_1, x_2, x_3, x_4) \in \mathbf{F}^4 : x_1 = 5x_2 \text{ and } x_3 = 7x_4\}.$$

Prove that T is surjective.

Proof:

$\operatorname{null} T = \{(5x_2, x_2, 7x_4, x_4) \in \mathbb{F}^4 : x_2, x_4 \in \mathbb{F}\}$

$= \operatorname{span}\{(5,1,0,0), (0,0,7,1)\}$

$\Rightarrow \dim(\operatorname{null} T) = 2.$

And $\dim(\operatorname{range} T) = \dim(\mathbb{F}^4) - \dim(\operatorname{null} T) = 4-2=2$

since $\operatorname{range} T \subseteq \mathbb{F}^2$ with $\dim(\mathbb{F}^2) = 2$

we have $\operatorname{range} T = \mathbb{F}^2 \Rightarrow T$ is surjective.

#

Q3:

10 Suppose U is a subspace of V with $U \neq V$. Suppose $S \in \mathcal{L}(U, W)$ and $S \neq 0$ (which means that $Su \neq 0$ for some $u \in U$). Define $T : V \to W$ by

$$Tv = \begin{cases} Sv & \text{if } v \in U, \\ 0 & \text{if } v \in V \text{ and } v \notin U. \end{cases}$$

Prove that T is not a linear map on V.

Proof:

Take any vector $v_1 \notin U$, then $Tv_1 = 0$.

Note that the vector $u + v_1 \notin U \Rightarrow T(u+v_1) = 0$.

(otherwise, $\exists\ \tilde{u} \in U$, s.t. $v_1 = \tilde{u} - u$

U is a subspace $\Rightarrow v_1 \in U$ contradition)

But $Tu + Tv_1 = Su \neq 0$, so $T(u+v_1) \neq Tu + Tv_1$.

T is not linear.

#